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On the Behavior of Best Uniform Deviations

S. J. POREDA

Department of Mathematics, Clark University, Worcester, Massachusetts 01610

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1. INTRODUCTORY REMARKS AND DEFINITIONS

For a continuous complex-valued function f defined on a compact set E in the plane and for $n \in Z^+ = \{0, 1, 2, \ldots\}$, let $p_n(f, E)$ denote the polynomial of degree n of best uniform approximation to f on E and let

$$\rho_n(f, E) = \max_{z \in E} | f(z) - p_n(f, E)(z)| = \| f - p_n(f, E)\|_E .$$

Also, if f is not itself a polynomial of degree n on E, let

$$\mu_n(f, E)(z) = [f(z) - p_n(f, E)(z)]/\rho_n(f, E).$$

Since the functions $\{\mu_n(f, E)\}_{n=0}^{\infty}$ are all of (uniform) norm one on E, a question which naturally arises is whether or not these functions converge on E, and if they do (or don't), what this implies about the function f with respect to the set E. In particular, we shall restrict ourselves to the unit circle U. The following examples will serve to shed light on the above question in this special case.

2. EXAMPLES

Let $f(z) = 1/(z - a)$ and as above, let $U = \{| z | = 1\}$. There are two cases.

Case 1 [1]. If $| a | > 1$, then $\mu_n(f, U)(z) = e^{i\theta_n}(1 - \bar{a}z)/(z - a)$, where $\theta_n = -n \cdot \arg (a)$.

Case 2 [4]. If $| a | < 1$, then $\mu_n(f, U)(z) = (1 - \bar{a}z)/(z - a)$, for all n.

Thus we see that in Case 1., the sequence $\{\mu_n(f, U)\}_{n=0}^{\infty}$ does not converge on U while in Case 2., it does. On the basis of these examples one might guess that if f is not analytic for $|z| < 1$ then this sequence converges, and conversely. Also, one might ask if the total change in the argument of $\mu_n(f, U)(z)$ on U tends to ∞ as n increases if f is analytic for $|z| < 1$, and again conversely. Our first theorem partially answers the first of these two questions.

3. Main Results

THEOREM 1. *If $f(z)$ is analytic in the unit* disk $D = \{|z| < 1\}$ *and continuous in $\bar{D}$ and not itself a polynomial, then the sequence $\{\mu_n(f, U)\}_{n=0}^{\infty}$ does not converge uniformly on U.*

Proof. By our hypothesis the functions $\{\mu_n(f, U)\}_{n=0}^{\infty}$ are analytic in D and continuous in $\bar{D}$. Consequently, if this sequence of functions were to converge uniformly on U to some function $L(z)$, then this function would likewise be analytic in D and continuous in $\bar{D}$. Furthermore, by a theorem of Walsh [5, p. 36], the function $L(z)$ would be the uniform limit of polynomials on U and so for any ϵ, $0 < \epsilon < 1$, there exists an $N \in Z^+$ and polynomial q_N of degree N such that

$$\| L(z) - \mu_N(f, U)(z)\|_U < \epsilon/2 \qquad \text{and} \quad \| L(z) - q_N(z)\|_U < 1 - \epsilon.$$

It then follows that

$$\| \mu_N(f, U)(z) - q_N(z)\|_U < 1 - \epsilon/2, \text{ which implies:}$$

$$\| f(z) - p_N(f, U)(z) - \rho_N(f, U)\, q_N(z)\|_U < (1 - \epsilon/2)\, \rho_N(f, U),$$

contrary to the definition of $\rho_n(f, U)$ and our assumption that f is not a polynomial.

The following corollary follows by the method of proof of Theorem 1.

COROLLARY. *Suppose $f(z)$ is a function continuous on U and not itself a polynomial. If the sequence $\{\mu_n(f, U)(z)\}_{n=0}^{\infty}$ converges uniformly on U to a function $L(z)$ then $p_n(L, U) \equiv 0$ for all n.*

Using the above corollary we can further characterize the limit function (if it exists) of the sequence $\{\mu_n(f, U)\}_{n=0}^{\infty}$, but first we require the following:

LEMMA. *If $\Phi(z)$ is continuous on U and $p_n(\Phi, U) \equiv 0$ for all n, then $|\phi(z)| = K$ for all $z \in U$ and for some constant $K \geqslant 0$.*

Proof. Suppose for some $z_0 \in U$, $|\Phi(z_0)| < \|\Phi\|_U$. Then there exists an open connected arc $\Gamma \subset U$ such that for $z \in \Gamma$, $\Phi(z)| < \|\Phi\|_U - \epsilon$ for some $\epsilon > 0$. Now $U - \Gamma$ is a closed connected arc on which Φ is continuous, and so by [5, p, 36] there exists polynomial $q_N(z)$ of some degree N for which $\|\Phi(z) - q_N(z)\|_{U-\Gamma} < \|\Phi\|_U - \epsilon$. Choosing λ, $0 < \lambda < 1$, such that $\lambda \|q_N\|_U < \epsilon/2$, we have

$$\|\Phi - \lambda q_N\|_{U-\Gamma} < (1 - \lambda)\|\Phi\|_U + \lambda(\|\Phi\|_U - \epsilon) = \|\Phi\|_U - \lambda\epsilon,$$

and

$$\|\Phi - \lambda q_N\|_\Gamma \leqslant \|\Phi\|_\Gamma + \lambda \|q_N\|_U < \|\Phi\|_U - \epsilon/2.$$

Thus, $\|\Phi - \lambda q_N\|_U < \|\Phi\|_U$ and so $p_N(\Phi, U) \not\equiv 0$ contradicting our hypothesis.

THEOREM 2. *Suppose $f(z)$ is continuous on U and not itself a polynomial. If the sequence $\{\mu_n(f, U)\}_{n=0}^{\infty}$ converges uniformly on U to $L(z)$, then $|L(z)| = 1$ for all $z \in U$. Furthermore, if $f(z)$ is in addition meromorphic in $\bar{D}$ then $L(z)$ is of the form*

$$L(z) = e^{i\theta} z^t \prod_{j=1}^{k} [(1 - \bar{a}_j z)/(z - a_j)],$$

where $t \in Z^+$, $|a_j| \neq 1$ for $j = 1, 2,..., k$, and where $|a_j| < 1$ for at least one j.

Proof. By the corollary to Theorem 1, $p_n(L, U) \equiv 0$ for all $n \in Z^+$, and so the first part of our theorem follows from the previous lemma. If, in addition, $f(z)$ is meromorphic in $\bar{D}$ then so is $L(z)$ and thus $L(z)$ cannot be analytic in $\bar{D}$ for otherwise this would mean that for some n, $p_n(L, U) \not\equiv 0$. Finally, one can easily show, using Róuche's Theorem, for instance, that a function meromorphic in $\bar{D}$ and having constant modulus one on U must be of the desired form.

4. REMARKS

Whether or not the converse of Theorem 1 is true is still open as is the question relating the behavior of the argument of $\mu_n(f, U)$ with the analyticity of f inside U. However, Theorem 2 suggests that the long-standing result of Caratheodory and Fejér [3] (also see [2]) concerning k-fold restricted polynomials may be generalized to include best approximation to rational functions.

References

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